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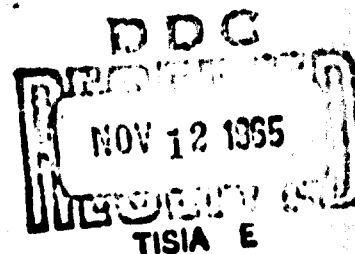
# EXPERIMENTAL METHODS FOR DETERMINING SHEAR MODULUS OF FIBER REINFORCED COMPOSITE MATERIALS

J. M. HENNESSEY  
JAMES M. WHITNEY  
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AIR FORCE MATERIALS LABORATORY  
RESEARCH AND TECHNOLOGY DIVISION  
AIR FORCE SYSTEMS COMMAND  
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

# **EXPERIMENTAL METHODS FOR DETERMINING SHEAR MODULUS OF FIBER REINFORCED COMPOSITE MATERIALS**

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## FOREWORD

This report was prepared by the Plastics and Composites Branch, Nonmetallic Materials Division and was initiated under Project No. 7340, "Nonmetallic and Composite Materials," Task No. 734003 "Structural Plastics and Composites." The work was administered under the direction of the Air Force Materials Laboratory, Research and Technology Division, with J. M. Mennessay, James M. Whitney, and Captain M. B. Riley as the project engineers.

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ABSTRACT

The shear modulus of elasticity of a fiber reinforced composite is an extremely important mechanical property. The problem of experimentally determining appropriate shear modulus for orthotropic materials such as fiber reinforced composites is not as simple and straightforward as in isotropic materials, and care must be taken both in the experimental methods and deviation and use of the appropriate equations.

This report presents four acceptable methods for determining the shear modulus of orthotropic materials. Two methods are appropriate for determining in-plane shear modulus and two others are for determining "bending" shear modulus which is applicable in calculating buckling loads of laminated plates and shells. A discussion of each method precedes the method of experimental determination. The derivation of all pertinent equations are presented in the appendix for easy reference.

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# SYMBOLS

|                 |   |   |
|-----------------|---|---|
| $\sigma_{ii}$   | = | Normal stress acting in the i direction                         |
| $\tau_{ij}$     | = | Shear stress acting on i-j plane                                |
| $\epsilon_{ii}$ | = | Normal shear strain in the i direction                          |
| $\gamma_{ij}$   | = | Shear strain in the i-j plane                                   |
| E               | = | Tensile modulus   |
| u               | = | Displacement in the x direction for Cartesian coordinate system |
| v               | = | Displacement in the y direction for Cartesian coordinate system |
| w               | = | Displacement in the z direction for Cartesian coordinate system |
| $\mu$           | = | Poisson's ratio   |

## INTRODUCTION

In determining the usefulness of a composite material as a structural member, it becomes extremely important to know its behavior under shear stresses. This means that shear strength and shear modulus must be determined. This report will be concerned with methods for experimentally measuring shear modulus.

If a flat plate is subjected to a pure shear stress  $\theta$  as shown in Figure 1, it will undergo a shear strain  $\gamma$ , defined by the angle  $\theta$ . Thus  $\gamma = \theta$ .

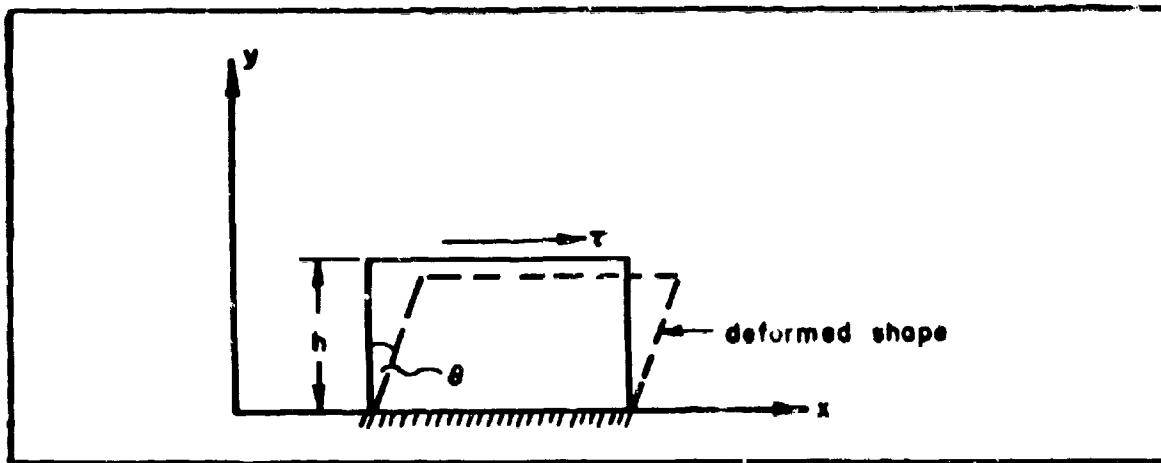


Figure 1. Flat Plate Subjected to Shear Stress

In the case of a perfectly elastic body Hooke's law tells us that  $\tau$  is directly proportional to  $\gamma$ . The proportionality constant  $G$  is the shear modulus or as it is often called the "modulus of rigidity." Thus it is a measure of a body's resistance to deformation due to shear stresses being applied.

Laminated structures present a rather unique situation because  $G$  may take on two distinct values depending on the application of the shear stresses. If a laminate of  $n$  layers is subjected to shear stress as shown in Figure 1, the deformation will occur in the  $x$ - $y$  plane and its shear modulus may be determined from the shear modulus of each layer by using the following formula:

$$G_s t = \sum_{i=1}^n G_i t_i \quad (1)$$

where  $G_i$  and  $t_i$  are the shear modulus and thickness respectively of each individual layer. The total thickness of the panel is denoted by  $t$ .  $G_s$  denotes the in-plane shear modulus.

If a laminate is subjected to twisting, it will undergo shear deformation which is three dimensional, that is, outside the  $x$ - $y$  plane. The rigidity or resistance to twisting of the laminate now becomes a function of the moment of inertia of the cross section just as the stiffness of a beam subjected to bending is a function of the moment of inertia. Thus the formula for  $G$  as a function of the properties in each individual layer becomes:

$$GI = \sum_{i=1}^n G_i I_i \quad (2)$$

where  $I_i$  is the moment of inertia of the cross section of each individual layer with respect to the centroidal axis of the laminate cross section and  $I$  is the moment of inertia of the entire cross section with respect to the same axis.  $G$  denotes the "bending" shear modulus which is applicable in the calculation of buckling loads of laminated plates and sheets. Thus we can see from Equations 1 and 2 there is a numerical difference which occurs in the case of laminates and not in the case of a one layer composite with unidirectional fibers.

Two of the experimental methods reported will involve twisting (three dimensional deformation) of the test sample and will therefore determine the twisting modulus of rigidity. Equations will be derived which will allow the modulus of rigidity, for in plane deformation, to be calculated from tensile modulus at  $0^\circ$ ,  $45^\circ$ , and  $90^\circ$  and Poisson's ratio. In-plane shear tests are extremely difficult to perform. The test method described in this report which deals with in-plane shear modulus has limitations which will be discussed. The four methods of shear modulus determination described have all been used with good results. The equations used in the calculations have been completely derived for easy reference and appear in the appendix.

It should be noted that all methods presented in this report determine shear modulus due to panel shear as opposed to interlaminar shear. This is illustrated in Figure 2 for the case of a laminated structure in the form of a flat plate. Panel shear involves shear modulus

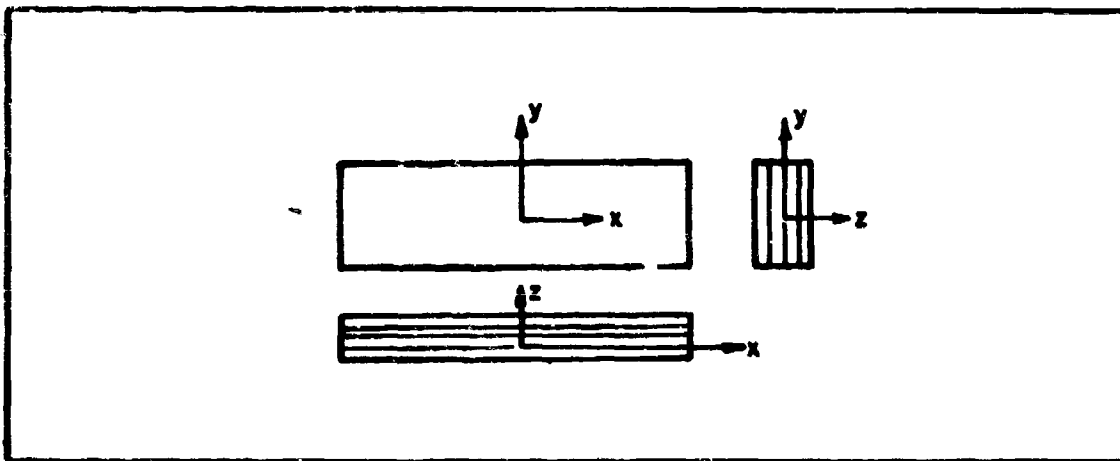


Figure 2. Laminated Flat Plate

relating shear stress to shear strain occurring in the x-y plane. Interlaminar shear involves shear modulus relating shear stress to shear strain in either the x-z or y-z plane.

## CALCULATIONS OF IN-PLANE SHEAR MODULUS

The shear modulus of an orthotropic system exhibiting Hookian behavior may be obtained by knowing the moduli of elasticity along the natural axis, Poisson's ratio and another elastic modulus at some angle to one of the natural axis.

Mathematically, we may derive this equation by use of the stress-strain relations for orthotropic plates. Then by use of transformation equations, we may derive equations for stress related to a new axis. A similar derivation may be obtained for strain. Relating the stress and strain equation an expression for shear modulus follows:

The equation, when  $\theta = 45^\circ$  is:

$$G_s = \frac{1}{\frac{4}{E_{45}} - \frac{1}{E_{22}} - \frac{1}{E_{11}} (1 - 2\mu_{12})} \quad (3)$$

Where:

$G_s$  = in-plane shear modulus

$E_{45}$  = modulus of Elasticity at  $45^\circ$  to the natural axis

$E_{11}$  = modulus of Elasticity along a natural axis

$\mu_{12}$  = Poisson's Ratio - ratio of strain along the 1 axis to strain along the 2 axis

$E_{22}$  = modulus of Elasticity normal to  $E_{11}$

## EXPERIMENTAL DETERMINATION OF IN-PLANE SHEAR MODULUS

### GENERAL DESCRIPTION

The in-plane shear modulus can be determined experimentally by deforming a square panel into a rhombic shape as shown in Figure 3. This can be achieved by several methods. The square panel can be forced directly to assume a rhombic shape. The difficulty of this method is keeping the panel flat during the test. Any distortion occurring will lead to error. Another method is the picture frame test in which the specimen is placed into a square straining frame with pinned corners. The frame (specimen) is deformed into a rhombic shape as shown in Figure 4. A third method is an edge flexure (that is, the specimen is oriented so that the laminations are distributed through the width rather than through the thickness) test as shown in Figure 5. Since this is a two point load application, the portion of the beam between the load points deflects as a function of bending only. But the deflection as measured from the support to the mid-span is additive, and therefore, the deflection due to shear can be determined by measuring the deflection of both the quarter and mid-points. Using the equations for deflection

$$y_1 = \frac{Px^3}{6EI} + 3\left(\frac{1}{2AG} + \frac{L^2}{32EI}\right)Px \quad 0 \leq x \leq \frac{L}{4} \quad (4)$$

$$y_2 = -\frac{PLx^2}{8EI} + \frac{PLx}{8EI} - \frac{PL^2}{384EI} + \frac{3PL}{8AG} \cdot \frac{L}{4} \leq x \leq \frac{L}{2} \quad (5)$$

and substituting distances to the measuring points, the in-plane shear modulus can be solved with simultaneous equations. The result is:

$$G_s = \frac{9PL}{8A \left( 11y_{\frac{L}{4}} - 8y_{\frac{L}{2}} \right)} \quad (6)$$

Plastic laminates reinforced with parallel fiber or cloth layers are orthotropic, that is, three mutually perpendicular axes of symmetry. As previously mentioned, the in-plane shear modulus can be obtained in the edgewise flexure test.

### OUTLINE OF TEST PROCEDURES

A bar of rectangular cross-section is tested in edgewise flexure as a simple beam, the bar resting on two supports. The span should be at least ten times the depth of the specimen. The load is applied at the two quarter points  $L/4$  and  $3L/4$ .

#### Apparatus

(a) Testing Machine -- a properly calibrated testing machine which can be operated at constant rates of crosshead motion over the range indicated, and the error in the load indicating system shall not exceed  $\pm 1$  percent. It shall be equipped with a deflection measuring device. The stiffness of the testing machine shall be such that the total elastic deformation of the

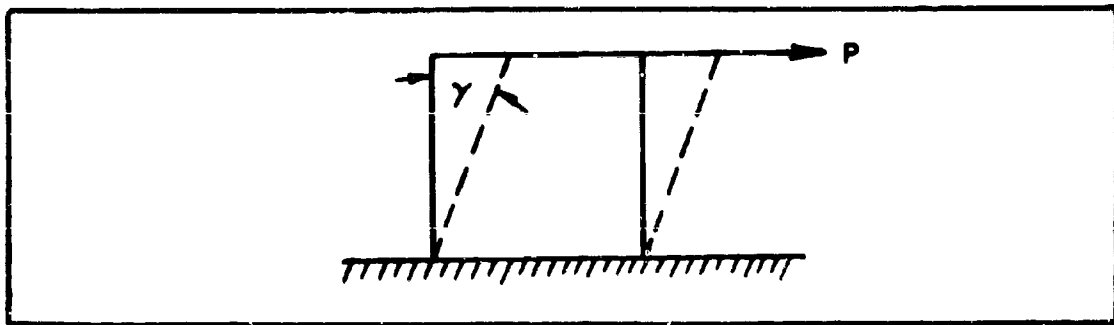


Figure 3. Forcing Square Panel Into Rhombic

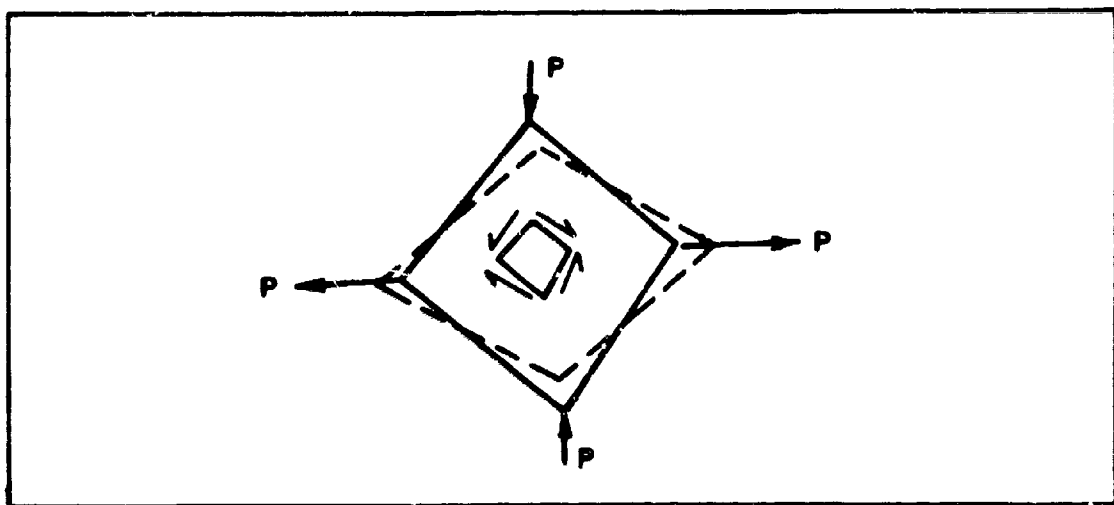


Figure 4. Picture Frame Test

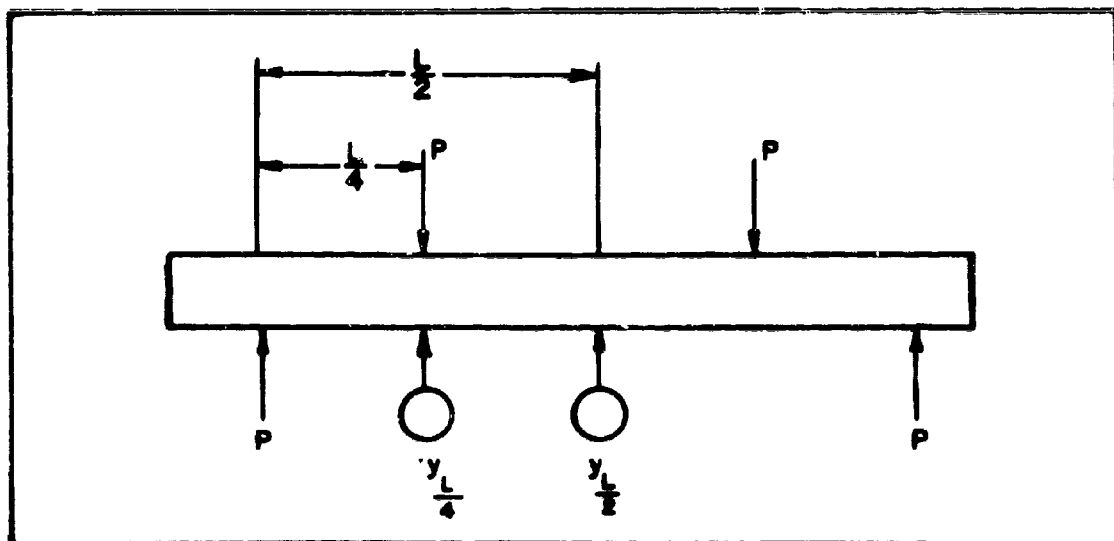


Figure 5. Quarter Point Edgewise Flexure

system does not exceed 1 percent of the total deflection of the test specimen during test or appropriate corrections shall be made. The load indicating mechanism shall be essentially free from inertia lag at the crosshead rate used.

(b) Loading Nose and Supports — the loading nose and supports shall have cylindrical surfaces. In order to avoid excessive indentation, the radius of the nose and supports shall be at least 3.2 mm (1/8 in.) for all specimens.

#### Test Specimens

The specimens may be cut from sheets, plates, or molded shapes, or may be molded to the desired finished dimensions.

For Materials 1.6 mm (1/16 in.) or Greater in Thickness — for edgewise tests the width of the specimen shall be the thickness of the sheet and the depth shall not exceed the width (Notes 1 and 2\*). For all tests the space shall be at least 10 times the depth of the beam. The specimen shall be long enough to allow for overhanging on each end of at least 10 percent of the span, but in no case less than 6.4 mm (1/4 in.) on each end. Overhang shall be sufficient to prevent the specimen from slipping through the supports.

#### Number of Test Specimens

(a) At least five specimens shall be tested for each sample in the case of isotropic materials or molded specimens.

(b) For each sample of anisotropic material in sheet form, at least five specimens shall be tested for each condition. Recommended conditions are flatwise and edgewise tests on specimens cut in lengthwise and crosswise directions of the sheet. For purposes of this test, "lengthwise" shall designate the principle axis of anisotropy, and shall be interpreted to mean the direction of the sheet known to be the stronger in flexure. "Crosswise" shall be the sheet direction known to be the weaker in flexure, and shall be at 90 degrees to the lengthwise direction.

#### Conditioning Test Specimens

Unless otherwise indicated in Material Specifications, all test specimens shall be conditioned in accordance with Procedure A described in Section 4(a) of the Methods of Conditioning Plastics and Electrical Insulating Materials for Testing (ASTM Designation: D 618)<sup>3</sup>, and tests shall be conducted in the Standard Laboratory Atmosphere as defined in the same specification.

\*Note 1 — Whenever possible, the original surfaces of the sheet shall be unaltered. However, where machine limitations make it impossible to follow the above criterion on the unaltered sheet, both surfaces shall be machined to the desired dimensions and the location of the specimens with reference to the total thickness shall be noted. The values obtained on specimens with machined surfaces may differ from those obtained on specimens with original surfaces. Consequently, any specifications for flexural properties on the thicker sheets must state whether the original surfaces are to be retained or not.

\*Note 2 — Edgewise tests are not applicable for sheets that are so thin that specimens meeting these requirements cannot be cut. If specimen depth exceeds the width, buckling may occur.

### Procedure

(a) Use an untested specimen for each measurement. Measure the width and thickness of the specimen to the nearest 0.03 mm (0.001 in.) at the center of the span. For specimens less than 2.54 mm (0.100 in.) in thickness, measure the thickness to the nearest 0.003 mm (0.0001 in.).

(b) Determine the span to be used as described in Section 5. After the span is set, measure the actual span length to the nearest 1 percent.

(c) Where Table 1 is used, set the machine for the specific rate of crosshead motion, or as near as possible to it.

(d) Align the loading nose and supports so that the axes of the cylindrical surfaces are parallel and the loading nose is midway between the supports. This parallelism may be checked by means of a plate with parallel grooves into which the loading nose and supports will fit when properly aligned. Center the specimen on the supports, with the long axis of the specimen perpendicular to the loading noses and supports.

(e) Apply the load to the specimen at the specified crosshead rate, and take simultaneous load-deflection data. Deflection shall be measured either by gages under the specimen in contact with it at the center of the span, and at one quarter point, the gages being mounted stationary relative to the specimen support, or by measurement of the motion of the loading nose relative to the supports with a gage at mid-span. In either case, appropriate corrections for indentation in the specimens and deflections in the weighing system of the machine shall be made. Load-deflection curves may be plotted to determine shear modulus.

TABLE 1  
SUGGESTED DIMENSIONS FOR TEST SPECIMENS

| Nominal Specimen Thickness, mm (in.) | Width of Specimen, mm (in.) | Length of Specimen, mm (in.) | Test Span, mm (in.) | Rate of Cross-head Motion mm (in.) per min. |
|--------------------------------------|-----------------------------|------------------------------|---------------------|---|
| 0.8 (1/32)                           | 25.4 (1)                    | 50.8 (2)                     | 15.9 (5/8)          | 0.51 (0.02)                                 |
| 1.6 (1/16)                           | 25.4 (1)                    | 50.8 (2)                     | 25.4 (1)            | 0.76 (0.03)                                 |
| 2.4 (3/32)                           | 25.4 (1)                    | 63.5 (2 1/2)                 | 38.1 (1 1/2)        | 1.02 (0.04)                                 |
| 3.2 (1/8)                            | 25.4 (1)                    | 76.2 (3)                     | 50.8 (2)            | 1.27 (0.05)                                 |
| 4.8 (3/16)                           | 12.7 (1/2)                  | 102 (4)                      | 76.2 (3)            | 2.03 (0.08)                                 |
| 6.4 (1/4)                            | 12.7 (1/2)                  | 127 (5)                      | 102 (4)             | 2.79 (0.11)                                 |
| 9.6 (3/8)                            | 12.7 (1/2)                  | 191 (7 1/2)                  | 152 (6)             | 4.06 (0.16)                                 |
| 12.7 (1/2)                           | 12.7 (1/2)                  | 254 (10)                     | 203 (8)             | 5.33 (0.21)                                 |
| 19.1 (3/4)                           | 19.1 (3/4)                  | 381 (15)                     | 305 (12)            | 8.13 (0.32)                                 |
| 25.4 (1)                             | 25.4 (1)                    | 495 (19 1/2)                 | 406 (16)            | 10.92 (0.43)                                |

**Retests**

Values for properties of shear modulus shall not be calculated for any specimen that breaks or buckles at some obvious, fortuitous flaw, unless such flaws constitute a variable being studied. Retests shall be made for any specimens on which values are not calculated.

**Calculation of Shear Modulus**

A beam is tested in edgewise flexure as a simple beam supported at two points and loaded at the quarter points. The shear modulus in the plane of the panel can be calculated with the following equation.

$$G_s = \frac{9PL}{8A \left( 11 y_{\frac{L}{4}} - 8 y_{\frac{L}{2}} \right)} \quad (7)$$

where:

$G$  = in-plane shear modulus

$y_{L/4}$  = deflection at the quarter point in millimeters (inches)

$y_{L/2}$  = deflection at mid-span

Note that this equation is based on the difference between two deflections. Thus extreme care must be exercised in making these measurements or the results will not be accurate.

$A$  =  $bd$ , the cross sectional area in square centimeters (square inches)

$P$  = load at a given point on the load deflection curve, in kilograms (pounds)

$L$  = span, in centimeters (inches)

$b$  = width of beam tested, in centimeters (inches) and

$d$  = depth of beam tested, in centimeters (inches)

**Arithmetic Mean** — for each series of tests, the arithmetic mean of all values obtained shall be calculated to three significant figures and reported as the "average value" for the particular property in question.

**Standard Deviation** — the standard deviation (estimated) shall be calculated as follows and reported to two significant figures:

$$S = \sqrt{\frac{\sum x^2 - n \bar{x}^2}{n - 1}} \quad (8)$$

where:

$S$  = estimated standard deviation

$X$  = value of single observation

$n$  = number of observations

$\bar{X}$  = arithmetic mean of the set of observations

Report — the report shall include the following

- (1) Complete identification of the material tested, including type, source, manufacturer's code number, form, principal dimensions, and previous history,
- (2) Directions of cutting and loading specimens,
- (3) Conditioning procedure,
- (4) Depth and width of the specimen,
- (5) Span length,
- (6) Span to depth ratio,
- (7) Radius of supports and loading nose,
- (8) Rate of crosshead motion in millimeters (inches) per minute,
- (9) Maximum strain in the outer fiber of the specimen,
- (10) Shear modulus, average and standard deviation.

The above outline is based on ASTM Designation: D 790-63, "Standard Method of Test for Flexural Properties of Plastics," and Military Standard 401A, "Sandwich Constructions and Core Materials; General Test Methods."

## FOUR POINT LOADING TEST

### GENERAL DESCRIPTION

The saddle test or four point loading is useful in determining one of the three possible shear moduli, where with the torsion test of rectangular beams, the effects of two of the three moduli are combined. In the saddle test, the shear modulus,  $G$ , is measured by imposing a pure twisting moment on a square plate. This is accomplished by placing four equal forces at the corners of the square plate. The forces are perpendicular to the plate with those forces at the first and third (diagonal) corners being upward and the other two forces downward. The corner loads force the square plate to assume a hyperbolic paraboloid or a saddle shaped surface. The sections of the surface made by vertical planes through the diagonals are two identical parabolas, one concave upward and the other concave downward, with the vertices at the center of the plate.

The shear modulus,  $G$ , can be computed from the ratio of the imposed loads and the vertical deflections of the plate with respect to the center of the plate under test. The derivation of the following equation is found in Appendix III.

$$G = \frac{3Pxy}{wt^3} \quad (9)$$

where:

$P$  = the load applied at each corner (lb)

$t$  = plate thickness (in)

$w$  = deflection of a point  $(x,y)$  on the diagonal with respect to the center of the plate (in)

$(x,y)$  = coordinates of the point on the diagonal (in) (See Figure 16)

$G$  = shear modulus with respect to the plane of the plate (psi)

The equation becomes:

$$G = \frac{3Pu^2}{2wt^3} \quad (10)$$

where:

$u$  = the diagonal distance from the center to point  $(x,y)$

To insure correct test results the following steps are necessary:

- (1) Never exceed one-half the proportional limit of the material.
- (2) It is recommended not to let  $u$  exceed half the distance from the center to the corner of the plate, thus avoiding any localized loading conditions at the corners of the plate.

(3) Since both the testing machine and the indicated deflection are twice the actual load at each corner and the deflections of one point with respect to the center, the indicated values may be substituted for  $P$  and  $w$ .

When possible, fairly thick specimens should be used in order to avoid effects associated with initial curvature. The side to thickness ratio should be maintained between 25:1 and 50:1. If any initial curvature is present, it can be minimized by obtaining two load-deflection curves, one with two given corners deflected downward and the second curve with these corners deflected upward. If there is a large difference in the two calculated shear moduli because of a deflection which is too large or lack of symmetry in their lay-up, the average of the moduli cannot be accepted as a good value of the actual shear modulus. In order to apply this test to orthotropic materials, the orthotropic axis must be parallel to the sides of the specimen (See Appendix III).

If the plate is deflecting according to theory, then lines drawn on the faces of the square test specimen, parallel to the edges, must remain straight during the experiment. A straight edge may be used to determine whether gross departures from the theoretical assumptions exist in any given plate. Also, the deflections of the four gage points with respect to the center should be approximately equal.

The shear modulus fixture is shown in Figure 6.

As mentioned before, localized loading effects may exist at the corners. To avoid this, the sharp loading points found in the fixture (Reference 1) may be replaced with roller bearings (micrometer ball attachments). The use of roller bearings at the loading points allows the plate to shift its points of contact as deflection occurs. The rounded surface also distributes force better than a pointed support which may indent the test plate. This indentation may cause separation of the fibers of a reinforced specimen as the load increases.

In order to eliminate reader error when recording load verses deflection, the dial indicator may be replaced with a microformer which allows use of the x-y recorder. This newer method may be necessary when testing low modulus materials, where the rate of deflection to load approaches the 0.3"/min crosshead speed.

## OUTLINE OF EXPERIMENTAL PROCEDURES

**Direction of Grain** — the orthotropic axis of the laminate shall be parallel and perpendicular to the edges of the test specimen.

**Test Specimen** — the test specimen shall be square, with the thickness equal to the thickness of the material and the length and width not less than 25 nor more than 40 times the thickness. The thickness, length, and width of each specimen shall be measured to an accuracy of not less than  $\pm 0.3$  percent. Care shall be taken to avoid obtaining test specimens with initial curvature.

**Loading Procedure** — the test specimen shall be supported on rounded supports having a radius of curvature not greater than one-half in. (6 mm) on opposite ends of a plate diagonal, and loaded in a similar manner on the opposite ends of the other diagonal. The loading and supporting frame shall be rigid. Figure 6 indicates the method of test and shows details of the plate shear on four point loading apparatus. The load shall be applied with a continuous and uniform motion of the plate corners to avoid the load and reaction effects. The plate shall not be stressed beyond its elastic range, and increments of load shall be chosen so that not less than 12 and preferable 15 load-deformation readings are taken.

To eliminate the effects of light initial curvature, two sets of data shall be obtained, the second set with the panel rotated 90 degrees about an axis through the center of the plate and



Figure 6. Shear Modulus Fixture

perpendicular to the plane of the plies. The two results shall be averaged to obtain the shear movable head at a rate of .003 times the length of the plate in inches (cm), expressed in inches (cm) per minute, with a permissible variation of  $\pm 25$  percent.

Deformation Measurements -- the deformation shall be measured to the nearest 0.001 in. (0.02 mm) at two points on each diagonal equidistant from the center of the plate. These measurements preferably shall be made at the quarter point of the diagonals, and if other points than these are chosen, care shall be taken to avoid location near the modulus for the plate. A satisfactory arrangement for measuring relative deformation is indicated in Figure 5.

Calculation -- the shearing modulus of elasticity shall be calculated as follows:

$$G = \frac{3u^2 P}{2t^3 w}$$

where:

G = shearing modulus in pounds per square inch (kg per sq. cm)

P = load applied by the test machine to the panel. (This load is twice that applied to each corner) in pounds (kg)

t = thickness of the plate in inches, (cm)

w = deflection as read on the dial gage (twice the deflection relative to the center) in inches (cm) (Note 3\*)

u = distance from the center of the panel to the point where the deflection is measured in inches (cm)

The above is based on ASTM Standards for the determination of plate shear.

\*Note 3 -- The average values of panel w are generally taken from the slope of a previously plotted load-deflection curve.

## TORSION TEST

### GENERAL DESCRIPTION

When a specimen is subjected to torsion, a couple normal to the axis of the member is developed in every cross-section. The formulas for circular shafts of constant cross-sections are:

$$\tau = \frac{Tc}{J} \quad (11)$$

$$G = \frac{TL}{\theta J} \quad (12)$$

These equations are based on the following assumptions: The resultant of external forces is a couple in the plane perpendicular to the axis of the shaft and these forces are in equilibrium. The specimen is of constant cross-section and during the test the axis remains straight. Planes normal to the axis remain plane and the diameter in any plane remains straight, that is, no warping occurs during the test.

The material must be homogeneous and isotropic. The stress must not exceed the proportional limit. While Equations (11) and (12) are used to obtain the torsional modulus of rupture, this value is not the true stress but rather a measure of the relative strengths of various materials.

The ordinary torsion formulas do not apply to a shaft having a cross-section other than circular. There are several disadvantages associated with noncircular cross-sections. St. Venant found that plane sections do not remain plane when subjected to torsion. As previously mentioned in the saddle shear test, three different principal shearing moduli are possible. These shearing stresses correspond to the shearing strains of the three planes of elastic symmetry. In the torsion test of non-circular cross-sections, these moduli have the combined effect of two of the three moduli. In order to obtain either modulus, it is necessary to combine results of tests on two or more specimens having different dimensions.

When testing rectangular cross-sections the above disadvantages can be eliminated through use of thin cross-sections. This will be shown in the appendix. As can be seen in the following diagram of the elemental block, the maximum shear stress in a rectangular bar is at the center of the long side, that is at a point on the surface nearest the longitudinal axis (axis of twist). St. Venant's paradox states if two bars of the same elastic material, but having different non-circular cross-sections, the one with a lower polar moment of inertia has the greater torsional stiffness and strength, provided there exists no concave surfaces in either bar. No shearing stresses can exist on the outside surfaces; therefore only the faces of the block elements not parallel to a surface are subjected to shearing stresses. Since no longitudinal stresses are present in block A, no counteracting stresses exist at the front and back faces. It is now apparent that the shearing stresses vary from a maximum at the center of the faces to zero at the corners. Equations (11) and (12) are modified in Appendix IV to allow for this parabolic stress pattern.

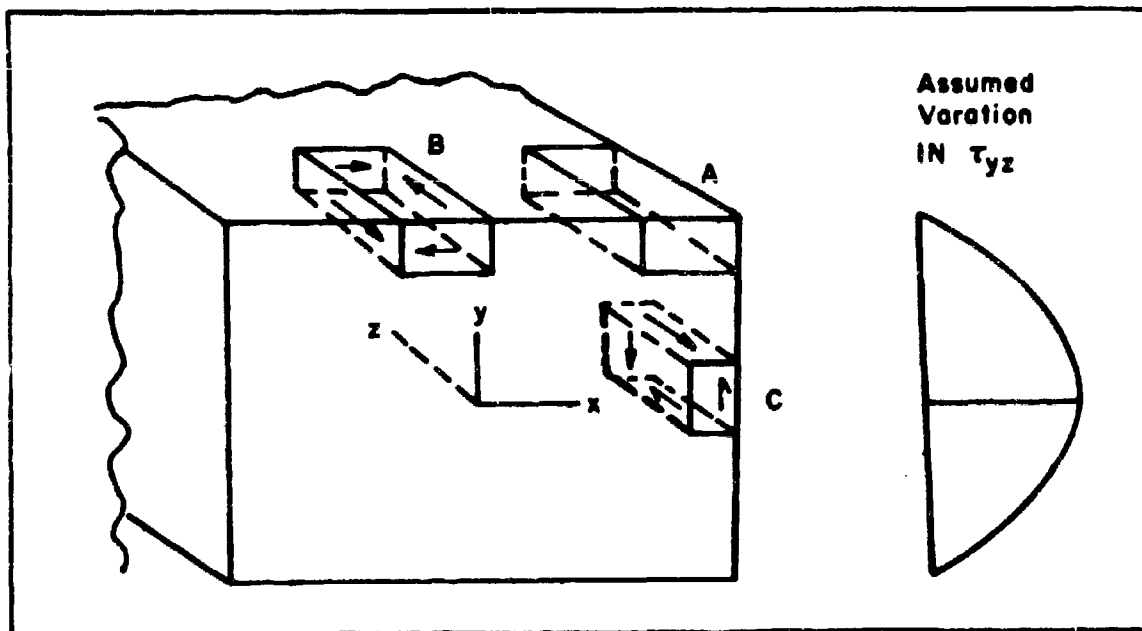


Figure 7. Torsion Member with Elemental Blocks

## OUTLINE OF EXPERIMENTAL PROCEDURES

This method describes a procedure for determining the stiffness characteristics of laminated plastics by direct measurement of the apparent modulus of rigidity.

**Significance** — the property actually measured by this method is the apparent modulus of rigidity,  $G$ , sometimes called the apparent shear modulus of elasticity. It is important to note that this property is not the same as the modulus of elasticity,  $E$ , measured in tension, flexure, or compression. The measured modulus of rigidity is termed "apparent" since it is the value obtained by measuring the angular deflection occurring when the specimen is subjected to an applied torque. Since the specimen may be deflected beyond its elastic limit the calculated value may not represent the true modulus of rigidity within the elastic limit of the material. In addition, the value obtained by this method will also be affected by the creep characteristics of the material, since the time of load application is arbitrarily fixed.

### Apparatus

(a) The testing machine must be capable of exerting a torque of approximately 0.12 to 1.2 Kg-cm (0.1 to 1.0 lb-in) on a test specimen with a span of 38.1 mm (1.5 in). A schematic diagram of a machine suitable for this test is shown in Figure 8. The amount of torque may be varied to suit the stiffness of the test specimen. Different weights should be available for this purpose. The actual amount of torque being applied by any given combination of weights, torque wheel radii, and shaft bearings shall be determined by calibration.

(b) Timer — a timer accurate to 0.1 sec.

(c) Micrometer — a dead-weight dial type micrometer capable of measuring thicknesses accurately to 0.0025 mm (0.0001 in.).

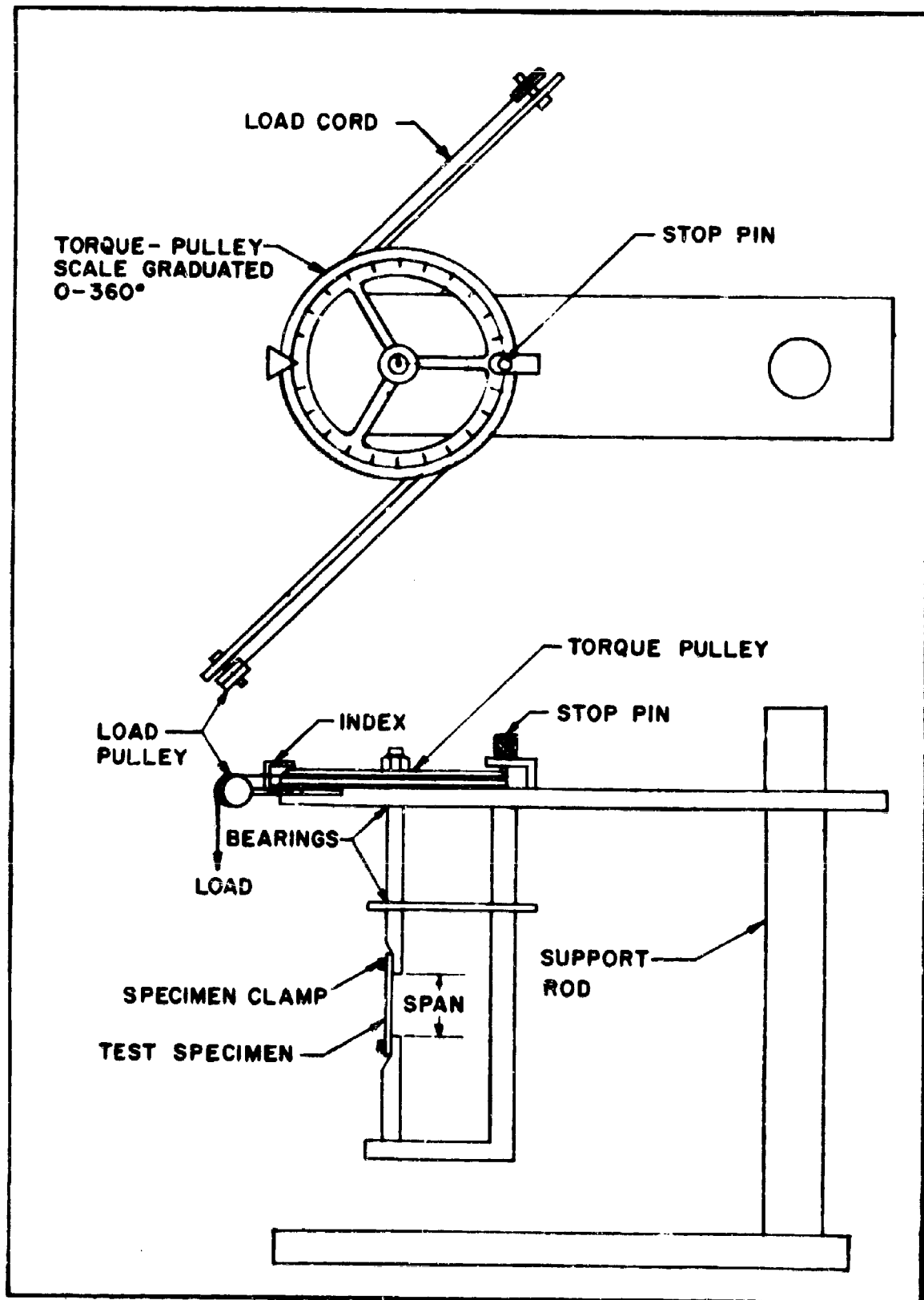


Figure 8. Torsion Tester

### Test Specimens

(a) Test specimens shall be of the shape and dimensions shown in Figure 9. They may be cut from compression-molded sheets. Care shall be taken to ensure that the test specimens are orthotropic with respect to the natural axis. A punch-and-jig unit is a useful tool for punching the mounting holes in the specimen.

(b) The thickness of cloth and angled fiber reinforced specimens shall not exceed one-tenth the width. For unwoven unidirectional fiber reinforced composites Equations 104 and accompanying table given in Appendix IV is applicable.

(c) Duplicate specimens of each material shall be tested.

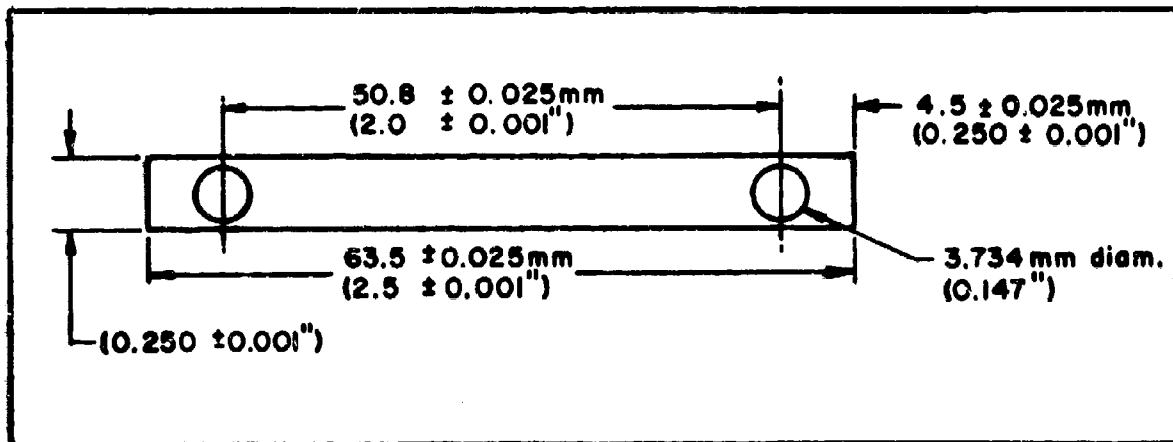


Figure 9. Test Specimen

### Procedure

- (a) Measure the width and thickness of the specimen to three significant figures.
- (b) Carefully mount the specimen in the apparatus. Adjust the clamps so there is no slack or tension in the specimen.
- (c) Release the torque pulley. After 5 seconds note the angular deflection of the pulley and return the torque pulley to its initial position. If the reading thus obtained does not fall within the range of 10 and 100 degrees of arc, vary the applied torque in such a way as to produce such a reading. (Note 4\*)
- (d) After each suitable reading is obtained, repeat the steps indicated in paragraph (c).

\*Note 4 — In order to obtain measured values of apparent modulus of rigidity,  $G$ , that are comparable to the true value of  $G$  it is desirable that measurements be made within the elastic limit of the material being tested. Therefore, torques shall be chosen that will cause deflections that are as small as it is practical to measure accurately on the machine being used.

Calculation

(a) Calculate the apparent modulus of rigidity, G, as follows

$$G = \frac{3T}{wt^3\theta} \quad (i3)$$

where;

T = Torque

w = Width ( $w \geq 10t$ )

t = Thickness

$\theta$  = Angle of twist

The above tests are based on ASTM Designation: D 1043-61T "Tentative Method of Test for Non-Rigid Plastics as a Function of Temperature by Means of a Torsion Test."

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## APPENDIX I

## DERIVATION OF ORTHOTROPIC SHEAR MODULUS EQUATION

In order to derive an equation for the shear modulus of a two dimensional orthotropic material as a function of Young's modulus of elasticity and Poisson's Ratio, we start by recalling the generalized equations of Hookian behavior for two dimensional anisotropy.

$$\epsilon_{xx} = a_{11} \sigma_{xx} + a_{12} \sigma_{yy} + a_{13} \tau_{xy} \quad (14)$$

$$\epsilon_{yy} = a_{12} \sigma_{xx} + a_{22} \sigma_{yy} + a_{23} \tau_{xy} \quad (15)$$

$$\gamma_{xy} = a_{13} \sigma_{xx} + a_{23} \sigma_{yy} + a_{33} \tau_{xy} \quad (16)$$

If we consider an orthotropic material whose elastic constants, parallel and perpendicular to the natural axes, may be measured, then, we may obtain stress-strain relations for orthotropic plates.

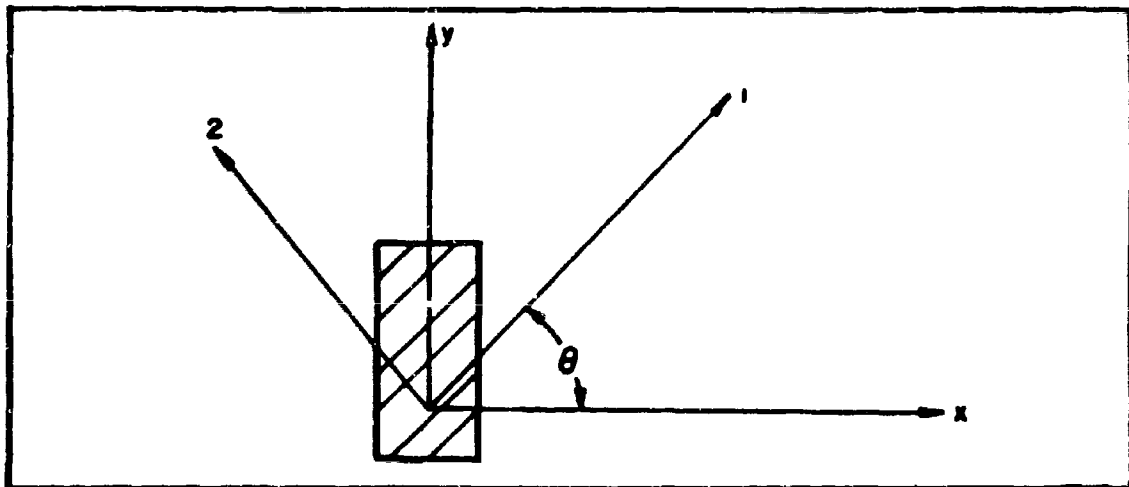


Figure 10. Axis System for Composite

In the system shown (Figure 10), the equations become:

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} \quad (17)$$

$$\epsilon_{11} = \frac{\sigma_{11}}{E_{11}} - \frac{\mu_{21} \sigma_{22}}{E_{22}} \text{ or } \frac{\sigma_{11}}{E_{11}} - \frac{\mu_{12} \sigma_{22}}{E_{11}} \quad (18)$$

$$\epsilon_{22} = \frac{\sigma_{22}}{E_{22}} - \frac{\mu_{12} \sigma_{11}}{E_{11}} \quad (19)$$

The laws of elasticity are known in directions 1 and 2 but not in the  $x$  and  $y$  direction. We desire to know the stresses as referred to a set of axes  $x$  and  $y$  which make an angle  $\theta$  with the natural axes, 1 and 2. Transformation equations for stress and strain may be derived from purely geometrical considerations for this new axis.

The following analysis derives the needed equations:

Consider a two-dimensional element (Figure 11) subjected to a general stress field,

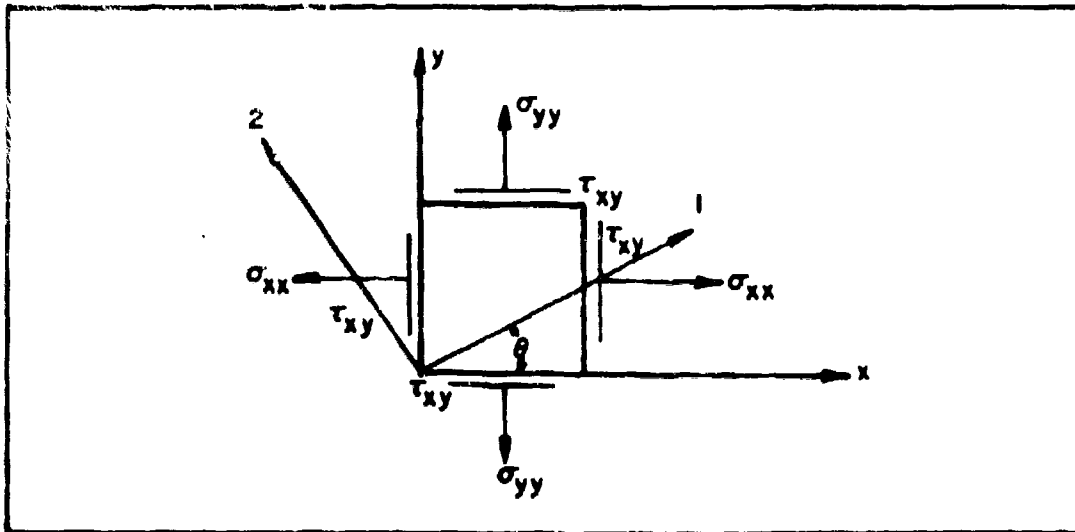


Figure 11. Two-Dimensional Element

If the stress distribution is known with respect to a given coordinate axis, then it can be determined with respect to any other coordinate axis by means of transformation equations.

So then consider an element of unit thickness with one face of unit length perpendicular to the 1 axis. All elements of a body in equilibrium are themselves in equilibrium if all external and internal forces are considered.

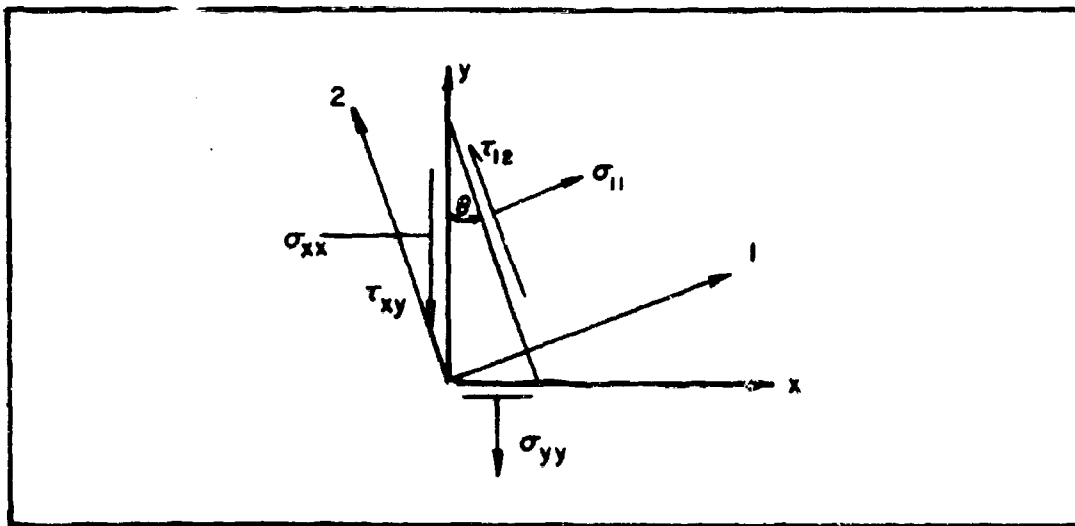


Figure 12. Stressed Orientation for Element

Summing forces in the 1 direction (Figure 12):

$$\begin{aligned} \sigma_{11} - \sigma_{xx} \cos \theta \cos \theta - \tau_{xy} \sin \theta \cos \theta - \\ \sigma_{yy} \sin \theta \sin \theta - \tau_{xy} \cos \theta \sin \theta = 0 \end{aligned}$$

or

$$\sigma_{11} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \quad (20)$$

Similarly, considering an element with face perpendicular to the 2 axis:

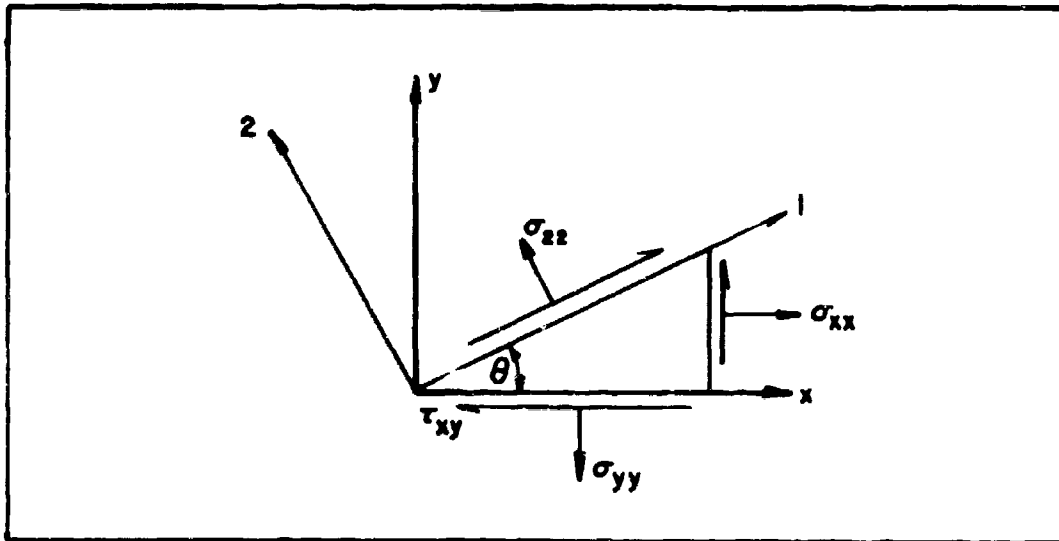


Figure 13. Stressed Orientation for Element

Summing forces in the 2 direction (Figure 13):

$$\begin{aligned} \sigma_{22} - \sigma_{xx} \sin \theta \sin \theta + \tau_{xy} \cos \theta \sin \theta \\ - \sigma_{22} \cos \theta \cos \theta + \tau_{xy} \sin \theta \cos \theta = 0 \end{aligned}$$

or

$$\sigma_{22} = \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta \quad (21)$$

Then summing forces on either element:

$$\begin{aligned} \tau_{12} + \sigma_{xx} \sin \theta \cos \theta - \tau_{xy} \cos \theta \cos \theta \\ - \sigma_{yy} \cos \theta \sin \theta + \tau_{xy} \sin \theta \sin \theta = 0 \end{aligned}$$

or

$$\tau_{12} = \sigma_{yy} \sin\theta \cos\theta - \sigma_{xx} \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta) \quad (22)$$

Summarily,

$$\sigma_{11} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2 \tau_{xy} \sin\theta \cos\theta \quad (23)$$

$$\sigma_{22} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2 \tau_{xy} \sin\theta \cos\theta \quad (24)$$

$$\tau_{12} = \sigma_{yy} \sin\theta \cos\theta - \sigma_{xx} \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta) \quad (25)$$

If we subject our system to a stress,  $\sigma_{xx}$ , then from Equations 14, 15, and 16, we may write

$$\epsilon_{xx} = a_{11} \sigma_{xx} \quad (26)$$

$$\epsilon_{yy} = a_{21} \sigma_{xx} \quad (27)$$

$$\gamma_{xy} = a_{31} \sigma_{xx} \quad (28)$$

Using Equation 23;

$$\epsilon_{xx} = a_{11} \sigma_{xx}$$

where:

$$a_{11} = \frac{1}{E_{xx}} \quad (29)$$

so that

$$E_{xx} = \frac{\sigma_{xx}}{\epsilon_{xx}} \quad (30)$$

We have derived equations for stress related to a new axis, we now need to obtain an equation for strain related to the new axis.

To set up strain equations at a point in a deformed body, consider a small length  $r$  along the 1 axis.

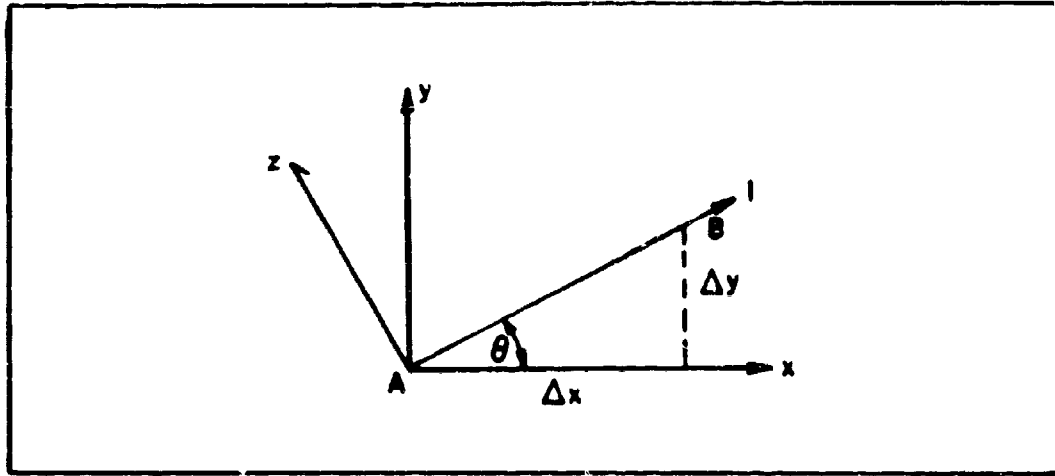


Figure 14. Axis for Strained Orientation

The projections of this length on the x and y axis are:

$$\Delta x = r \cos \theta \quad (31)$$

$$\Delta y = r \sin \theta \quad (32)$$

If  $u$  and  $v$  are displacements of the point A, the displacements of the point B can be expressed:

$$u' = u + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \quad (33)$$

$$v' = v + \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \quad (34)$$

After deformation, the coordinates of point B, which were originally  $x$  and  $y$  become:

$$\Delta x + u' - u = \Delta x + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \quad (35)$$

$$\Delta y + v' - v = \Delta y + \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \quad (36)$$

The new length of  $r$  after elongation becomes  $r + \epsilon r$ , therefore:

$$\begin{aligned} (r + \epsilon r)^2 &= \left( \Delta x + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right)^2 \\ &+ \left( \Delta y + \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right)^2 \end{aligned} \quad (37)$$

Dividing by  $r$  and substituting for  $x$  and  $y$

$$(1 + \epsilon)^2 = \left[ \cos\theta \left( 1 + \frac{\partial v}{\partial x} \right) + \sin\theta \frac{\partial v}{\partial y} \right]^2 + \left[ \cos\theta \left( \frac{\partial v}{\partial x} \right) + \sin\theta \left( 1 + \frac{\partial v}{\partial y} \right) \right]^2$$

Since  $\epsilon$  and derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

are small quantities, all orders higher than the first may be neglected.

Expand and neglect higher order terms:

$$1 + 2\epsilon = \cos^2\theta + 2\cos^2\theta \frac{\partial u}{\partial x} + 2\sin\theta\cos\theta \frac{\partial u}{\partial y} + \sin^2\theta + 2\sin^2\theta \frac{\partial v}{\partial y} + 2\sin\theta\cos\theta \frac{\partial v}{\partial x}$$

Then the unit strain through a unit area along AB becomes:

$$\epsilon_{11} = \cos^2\theta \frac{\partial u}{\partial x} + \sin\theta\cos\theta \frac{\partial u}{\partial y} + \sin^2\theta \frac{\partial v}{\partial y} + \sin\theta\cos\theta \frac{\partial v}{\partial x} \quad (38)$$

And since by the classical strain-displacement relations:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad (39)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \quad (40)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (41)$$

then

$$\epsilon_{11} = \epsilon_{xx} \cos^2\theta + \epsilon_{yy} \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \quad (42)$$

and conversely,

$$\epsilon_{xx} = \epsilon_{11} \cos^2 \theta + \epsilon_{22} \sin^2 \theta + \gamma_{12} \sin \theta \cos \theta \quad (43)$$

Then using Equation 30

$$\frac{1}{E_{xx}} = \frac{\epsilon_{xx}}{\sigma_{xx}}$$

and substituting Equation 43

$$\frac{1}{E_{xx}} = \frac{1}{\sigma_{xx}} \left[ \epsilon_{11} \cos^2 \theta + \epsilon_{22} \sin^2 \theta + \gamma_{12} \sin \theta \cos \theta \right]$$

and from Equation 43

$$\begin{aligned} \frac{1}{E_{xx}} = \frac{1}{\sigma_{xx}} & \left[ \left( \frac{\sigma_{11}}{E_{11}} - \frac{\mu_{12} \sigma_{22}}{E_{11}} \right) \cos^2 \theta + \left( \frac{\sigma_{22}}{E_{22}} - \frac{\mu_{12} \sigma_{11}}{E_{11}} \right) \sin^2 \theta \right. \\ & \left. + \frac{\tau_{12}}{G_{12}} \sin \theta \cos \theta \right] \end{aligned}$$

By using Equations 17, 18, and 19,

$$\begin{aligned} \frac{1}{E_{xx}} = \frac{1}{\sigma_{xx}} & \left[ \frac{\cos^2 \theta}{E_{11}} \left( \sigma_{xx} \cos^2 \theta - \mu_{12} \sigma_{xx} \sin^2 \theta \right) \right. \\ & \left. + \sin^2 \theta \left( \frac{\sigma_{xx} \sin^2 \theta}{E_{22}} - \frac{\mu_{12} \cos^2 \theta}{E_{11}} \right) + \frac{\sigma_{xx} \cos^2 \theta \sin^2 \theta}{G_{12}} \right] \\ \frac{1}{E_{xx}} = & \frac{\cos^4 \theta}{E_{11}} - \frac{2 \mu_{12} \sin^2 \theta \cos^2 \theta}{E_{11}} + \frac{\sin^4 \theta}{E_{22}} + \frac{\cos^2 \theta \sin^2 \theta}{G_{12}} \quad (44) \end{aligned}$$

Solving for  $G_{12}$  yields:

$$G_{12} = \frac{\cos^2 \theta \sin^2 \theta}{\frac{1}{E_{xx}} - \frac{\sin^4 \theta}{4E_{22}} - \frac{1}{E_{11}} \left( \cos^4 \theta - 2 \mu_{12} \sin^2 \theta \cos^2 \theta \right)} \quad (45)$$

When  $\theta = 45^\circ$

$$G_s = G_{12} = \frac{1}{\frac{4}{E_{45}} - \frac{1}{E_{22}} - \frac{1}{E_{11}}(1 - 2\mu_{12})} \quad (46)$$

## APPENDIX II

CALCULATION OF SHEAR MODULUS OF A FIBER  
REINFORCED COMPOSITE FROM THE DEFLECTION  
OF A QUARTER-POINT SYMMETRICALLY LOADED  
BEAM WITH A RECTANGULAR CROSS-SECTION

The effect of shearing force on the deflection of a beam has been discussed by Timoshenko (Reference 11). He shows that the slope of the deflection curve due to shear is equal to the shearing strain. Thus

$$\frac{dy_s}{dx} = \frac{\tau_{xy}|_{y=0}}{G} = \frac{VQ}{IwG} \quad (47)$$

where:

$y_s$  = deflection of beam due to shear

$V$  = shear force

$Q$  = first moment of area about the neutral axis

$I$  = moment of inertia

$w$  = width of the beam at  $y = 0$  (this is assuming the neutral axis is located at  $y = 0$ ).

Assuming small deflections, the equation for curvature of the deflected curve considering both shear and bending is

$$\frac{d^2 y}{dx^2} = \frac{d^2 y_s}{dx^2} + \frac{d^2 y_b}{dx^2} = -\frac{M}{EI} + \frac{Q}{Iw} \frac{dV}{dx} \quad (48)$$

where:

$y_b$  = deflection due to bending as obtained from the Euler-Bernoulli law.

Looking at Figure 15 we see a simply supported fiber reinforced beam loaded with a force  $P$  at  $x = L/4$  and at  $x = 3L/4$ . Since only the normal stress  $\sigma_{xx}$  and the shear stress  $\tau_{xy}$  are considered, simple beam theory will still apply to the case of an orthotropic material.

For a beam of rectangular cross-section

$$Q = \frac{wd^2}{8}, \quad I = \frac{wd^3}{12}$$

where  $d$  = depth of the beam. Equation 47 becomes

$$\frac{dy_s}{dx} = \frac{3}{2} \frac{V}{AG}$$

where  $A$ , of course, is the cross-section area. Using Equation 48 the equation for curvature will be

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} + \frac{3}{2AG} \frac{dV}{dx} \quad (49)$$

Using this equation for the section  $0 \leq x \leq L/4$  along with the shear and moment diagram we have

$$\frac{d^2y_1}{dx^2} = -\frac{Px}{EI} + \frac{3}{2AG} \frac{dP}{dx}$$

$$\frac{dy_1}{dx} = -\frac{Px^2}{2EI} + \frac{3P}{2AG} + c_1$$

$$y_1 = -\frac{Px^3}{6EI} + \frac{3Px}{2AG} + c_1x + c_2$$

at  $x = 0$ ,  $y_1 = 0$ ,  $\therefore c_2 = 0$

$$y_1 = \frac{Px^3}{6EI} + \frac{3Px}{2AG} + c_1x$$

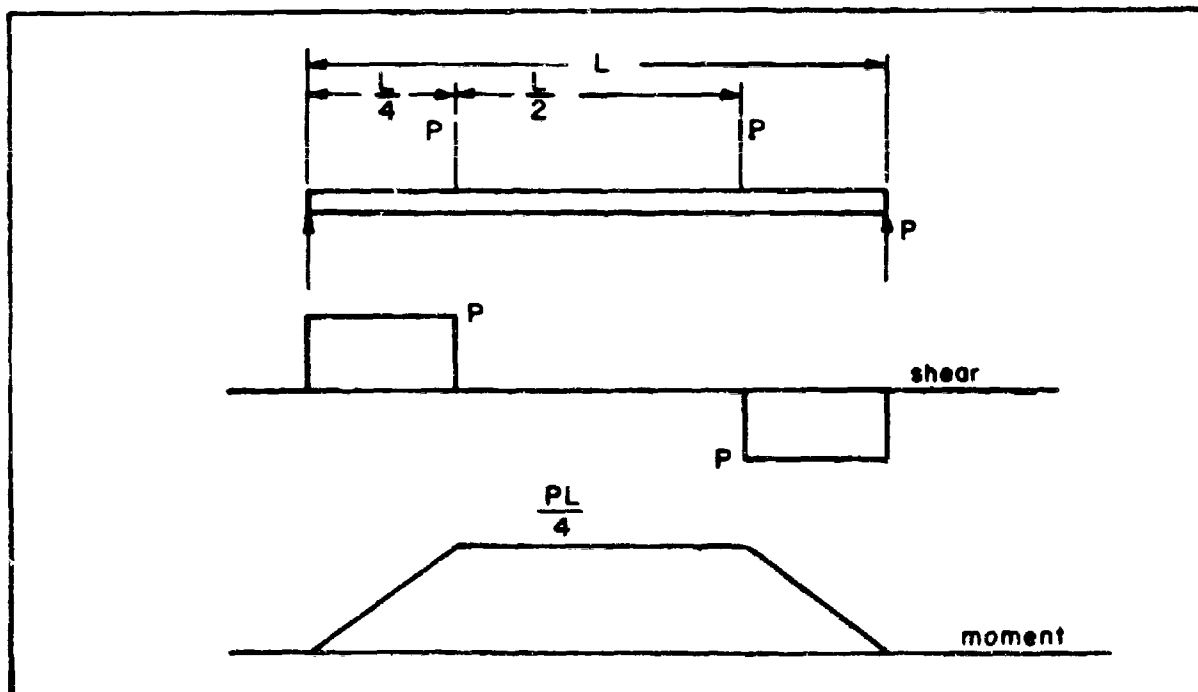


Figure 15. Simply Supported Fiber Reinforced Composite Beam with Quarter-Point Symmetric Loading Showing Shear and Moment Diagram

Again applying Equation 48 to the section  $L/4 \leq x \leq L/2$  using the shear and moment diagram we have

$$\frac{d^2 y_2}{dx^2} = - \frac{PL}{4EI}$$

$$\frac{dy_2}{dx} = - \frac{PLx}{4EI} + c_3$$

$$\text{at } x = \frac{L}{2}, \frac{dy_2}{dx} = 0$$

$$\therefore c_3 = \frac{PL^2}{8EI}$$

$$\frac{dy_2}{dx} = - \frac{PLx}{4EI} + \frac{PL^2}{8EI}$$

Integrating again

$$y_2 = - \frac{PLx^2}{8EI} + \frac{PL^2 x}{8EI} + c_4$$

To determine the constants  $c_1$  and  $c_4$  we look at the slope and deflection at the point  $x = L/4$ . The slope due to bending in the first section must be equal to the slope due to bending in the second section. Thus

$$\left( \frac{dy_1}{dx} - \frac{3P}{2AG} \right) = \left( \frac{dy_2}{dx} \right)$$

$$- \frac{PL^2}{32EI} + \frac{3P}{2AG} - \frac{3P}{2AG} + c_1 = - \frac{PL^2}{16EI} + \frac{PL^2}{8EI}$$

$$\therefore c_1 = \frac{3PL^2}{32EI}$$

$$y_1 = - \frac{Px^3}{6EI} + 3 \left( \frac{1}{2AG} + \frac{L^2}{32EI} \right) Px \quad (50)$$

Must also have continuity of deflection at  $x = \frac{L}{4}$ . Thus

$$\begin{aligned}
 -\frac{PL^3}{384EI} + \frac{3PL}{8AG} + \frac{3PL^3}{128EI} &= -\frac{PL^3}{128EI} + \frac{PL^3}{32EI} + c_4 \\
 \therefore c_4 &= -\frac{PL^3}{384EI} + \frac{3PL}{8AG} \\
 y_2 &= -\frac{PLx^3}{8EI} + \frac{PL^2x}{8EI} - \frac{PL^3}{384EI} + \frac{3PL}{8AG}
 \end{aligned} \tag{51}$$

The deflections at  $x = \frac{L}{4}$  and  $x = \frac{L}{2}$  are

$$\begin{aligned}
 y_{\frac{L}{4}} &= \frac{PL^3}{48EI} + \frac{3PL}{8AG} \\
 y_{\frac{L}{2}} &= \frac{11PL^3}{384EI} + \frac{3PL}{8AG}
 \end{aligned}$$

Multiplying the first of these equations by  $11/8$  and subtracting the second from it will yield

$$\begin{aligned}
 \frac{11}{8} \left( y_{\frac{L}{4}} - y_{\frac{L}{2}} \right) &= \frac{9PL}{64AG} \\
 G &= \frac{9PL}{8A \left( 11 y_{\frac{L}{4}} - 8 y_{\frac{L}{2}} \right)}
 \end{aligned} \tag{52}$$

## APPENDIX III

DERIVATION OF PLATE EQUATION FOR  
SADDLE SHEAR TEST

According to St. Venant's principle, a square plate loaded with a constant twisting moment along its edges is statically equivalent to the loading shown in Figure 16, provided the thickness of the plate is small in comparison to the other dimensions. For a more detailed justification of this statement see Reference 12.

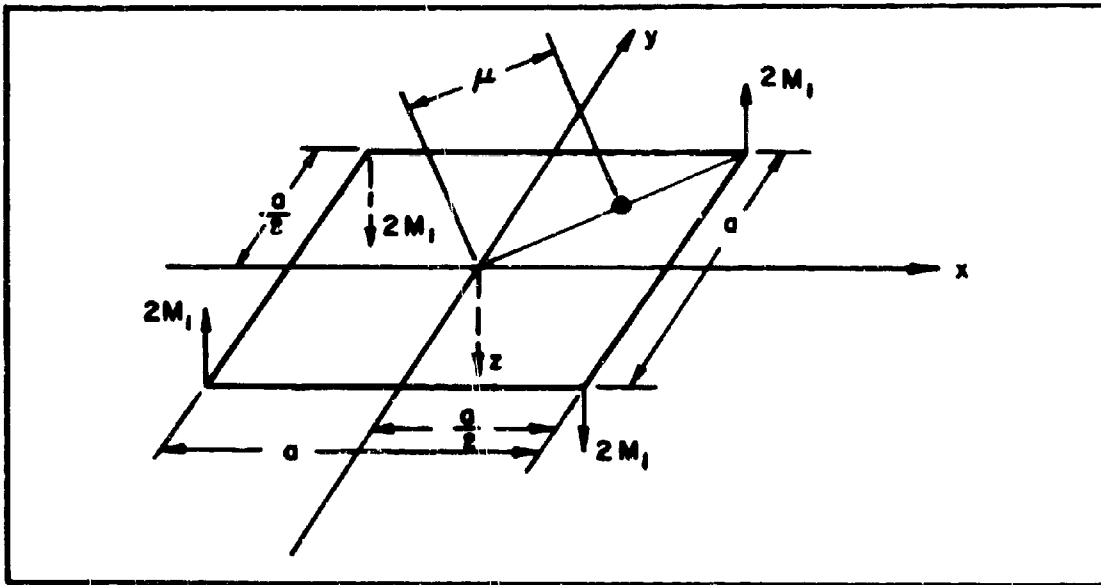


Figure 16. Orthotropic Plate Loaded at Corners

For orthotropic plates

$$G_{xx} = \frac{1}{E_x} (\sigma_{xx} - \mu_{xy} \sigma_{yy}) \quad (53)$$

$$e_{yy} = \frac{1}{E_y} (\sigma_{yy} - \mu_{yx} \sigma_{xx}) \quad (54)$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \quad (55)$$

Let  $w$  = deflection in  $z$  direction of any point  $(x,y)$  in the plane of the plate,

$$M_x = -D_1 \left( \frac{\partial^2 w}{\partial x^2} + \mu_{xy} \frac{\partial^2 w}{\partial y^2} \right) \quad (56)$$

$$M_y = -D_2 \left( \frac{\partial^2 w}{\partial y^2} + \mu_{yx} \frac{\partial^2 w}{\partial x^2} \right) \quad (57)$$

$$M_{xy} = \frac{Gt^3}{6} \frac{\partial^2 w}{\partial x \partial y} \quad (58)$$

where:

$M_x$  = moment per unit length along the edge parallel to the y-axis

$M_y$  = moment per unit length along the edge parallel to the x-axis.

$M_{xy}$  = twisting moment per unit length distributed along the edges.

$$D_1 = \frac{E_x t^3}{12(1 - \mu_{xy}^2)}, \quad D_2 = \frac{E_y t^3}{12(1 - \mu_{yx}^2)}$$

In this particular problem

$$M_x = M_y = 0; \quad M_{xy} = M_1$$

Using Equation 52

$$\frac{6M_1}{Gt^3} = \frac{\partial^2 w}{\partial x \partial y}$$

Integrating with respect to x

$$\frac{\partial w}{\partial y} = \frac{6M_1}{Gt^3} x + f_1'(y)$$

Integrating with respect to y

$$w = \frac{6M_1}{Gt^3} xy + f_1(y) + f_2(x) \quad (59)$$

from Equation 59

$$\frac{\partial^2 w}{\partial x^2} = f_2''(x)$$

$$\frac{\partial^2 w}{\partial y^2} = f_1''(y)$$

Using Equations 56 and 57 with the conditions  $M_x = M_y = 0$  will yield

$$\begin{aligned} f_2''(x) + \mu_{xy} f_1''(y) &= 0 \\ f_1''(y) + \mu_{yx} f_2''(x) &= 0 \end{aligned}$$

These equations must be 0 for all values of  $x$  and  $y$ .

$$\therefore f_2''(x) = c_1 = \text{constant} \quad (60)$$

$$f_1''(y) = c_2 = \text{constant} \quad (61)$$

$$\begin{vmatrix} 1 & \mu_{xy} \\ 1 & \mu_{yx} \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

We see the solution to this is  $c_1 = c_2 = 0$ . Using Equations 60 and 61:

$$f_2''(x) = 0 \quad (62)$$

$$\therefore f_2(x) = c_3 x + c_4$$

$$f_1''(y) = 0 \quad (63)$$

$$\therefore f_1(y) = c_5 y + c_6$$

Using Equations 62 and 63 in Equation 68 we have

$$w = \frac{6M_1}{Gt^3} xy + c_3 x + c_5 y + c_7$$

where:  $c_7 = c_4 + c_6$

Other conditions

$$1. \quad w(0,0) = 0 \quad \therefore c_7 = 0$$

$$2. \quad \left. \frac{\partial w}{\partial x} \right|_{x=y=0} = 0 \quad \therefore c_3 = 0$$

$$3. \quad \left. \frac{\partial w}{\partial y} \right|_{x=y=0} = 0 \quad \therefore c_5 = 0$$

Now

$$w = \frac{GM_1}{Gt^3} xy$$

Replacing  $M_1$  by the statically equivalent load shown in Figure 16 we have

$$w = \frac{3P}{Gt^3} xy \quad (64)$$

where  $P = 2M_1$

If we let  $u$  = the distance from the origin along a diagonal of the plate:

$$x = y = \frac{u}{\sqrt{2}}$$

$$\therefore w = \frac{3Pu^2}{2Gt^3} \quad (65)$$

**\*Note 5** — The orthotropic stress-strain relationships are good only when the  $x$  and  $y$  axis are both principle axis of symmetry of the orthotropic material. Otherwise Equations 64 and 65 will not work.

## APPENDIX IV

## TORSION OF AN ORTHOTROPIC BAR

The St. Venant semi-inverse approach will be used.

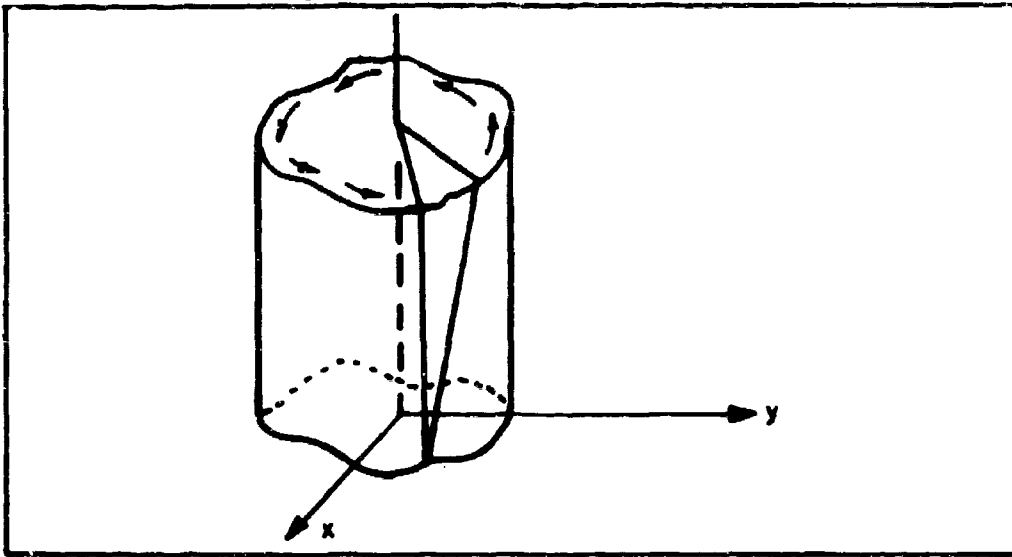


Figure 17. Bar of Arbitrary Cross Section Subjected to Torsion

Assume

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0 \quad (66)$$

Now applying orthotropic stress-strain relationships (assuming x - y axis to be orthotropic axis of material)

$$\left. \begin{aligned} \epsilon_{xx} &= \frac{1}{E_{xx}} (\sigma_{xx} - \mu_{xy} \sigma_{yy} - \mu_{xz} \sigma_{zz}); \\ \epsilon_{yy} &= \frac{1}{E_{yy}} (\sigma_{yy} - \mu_{yx} \sigma_{xx} - \mu_{yz} \sigma_{zz}); \\ \epsilon_{zz} &= \frac{1}{E_{zz}} (\sigma_{zz} - \mu_{zx} \sigma_{xx} - \mu_{zy} \sigma_{yy}); \\ \gamma_{xy} &= \frac{1}{G_{xy}} \tau_{xy}; \gamma_{yz} = \frac{1}{G_{yz}} \tau_{yz}; \gamma_{xz} = \frac{1}{G_{xz}} \tau_{xz} \end{aligned} \right\} \quad (67)$$

we see that

$$\left. \begin{aligned}
 \epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \gamma_{xy} &= 0 \\
 \epsilon_{xx} = \frac{\partial u}{\partial x} = 0 &\longrightarrow u = f_1(y, z) \\
 \epsilon_{yy} = \frac{\partial v}{\partial y} = 0 &\longrightarrow v = f_2(x, z) \\
 \epsilon_{zz} = \frac{\partial w}{\partial z} = 0 &\longrightarrow w = f_3(x, y) \\
 \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 &= \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}
 \end{aligned} \right\} \quad (68)$$

Looking at a cross-section of a bar of arbitrary cross-section we can determine the displacements

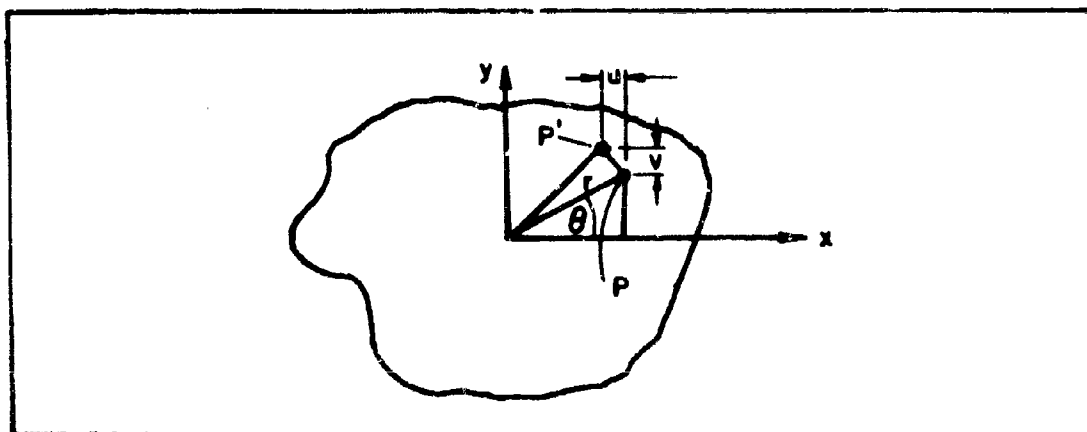


Figure 18. Cross-Section of Bar Having an Arbitrary Shape and Showing Displacements

P is a point at an angle  $\theta$  to the x axis and a distance r from the origin. P' is the same point after being rotated through an angle  $\theta_z$  where  $\theta$  is the angular twist per unit length. Assuming the displacements are small we have from Figure 18

$$\left. \begin{aligned}
 u &= -r\theta_z \sin \gamma = -y\theta_z \\
 v &= r\theta_z \cos \gamma = x\theta_z
 \end{aligned} \right\} \quad (69)$$

In order to solve an elasticity problem it is necessary to satisfy equilibrium, compatibility, and boundary conditions.

Looking first at the equilibrium equations

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \right\} \quad (70)$$

Equations 70 will be satisfied provided

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \longrightarrow \tau_{xz} = q_1(x, y) \quad (71)$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \longrightarrow \tau_{yz} = q_2(x, y) \quad (72)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad (73)$$

By using the stress strain relationships, Equations 67, and the strain displacement relationships, Equations 68, 71 and 72 are satisfied exactly. Since  $u$  and  $v$  are continuous functions they will satisfy compatibility.

In this approach a stress function  $\phi(x, y)$  is used which will satisfy equilibrium. If we pick  $\phi$  such that

$$\left. \begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} \end{aligned} \right\} \quad (74)$$

Equation 73 will be satisfied and thus the equilibrium conditions met.

Now looking at the compatibility equations

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} \quad (75)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial y \partial z} = \frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial \epsilon_{xz}}{\partial y^2} \quad (76)$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \epsilon_{xz}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2} \quad (77)$$

$$2 \frac{\partial^2 \epsilon_{xz}}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (78)$$

$$2 \frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y} \left( -\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (79)$$

$$2 \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (80)$$

it can be seen that since  $\phi$  is not a function of  $z$ , Equations 75, 76, 77, and 80 will be satisfied. Using Equations 78 and 79 along with stress strain relationships we have

$$\frac{\partial}{\partial x} \left( \frac{1}{G_{xy}} \frac{\partial \tau_{xz}}{\partial y} - \frac{1}{G_{yz}} \frac{\partial \tau_{yz}}{\partial x} \right) = 0$$

$$\frac{\partial}{\partial y} \left( \frac{1}{G_{yz}} \frac{\partial \tau_{yz}}{\partial x} - \frac{1}{G_{xz}} \frac{\partial \tau_{xz}}{\partial y} \right) = 0$$

Using  $\phi$  in these relationships yields

$$\frac{\partial}{\partial x} \left( \frac{1}{G_{yz}} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{G_{xz}} \frac{\partial^2 \phi}{\partial y^2} \right) = 0$$

$$\frac{\partial}{\partial y} \left( \frac{1}{G_{yz}} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{G_{xz}} \frac{\partial^2 \phi}{\partial y^2} \right) = 0$$

Thus

$$\frac{1}{G_{yz}} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{G_{xz}} \frac{\partial^2 \phi}{\partial y^2} = f(z) \quad \text{or a constant}$$

Using the displacements and the stress-strain relationships we have

$$\tau_{yz} = G_{yz} \left( \frac{\partial w}{\partial y} + x \theta \right)$$

$$\tau_{xz} = G_{xz} \left( \frac{\partial w}{\partial x} - y \theta \right)$$

or

$$\frac{\partial w}{\partial y} = \frac{\tau_{yz}}{G_{yz}} - x \theta \quad (81)$$

$$\frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G_{xz}} + y \theta \quad (82)$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{G_{xz}} \frac{\partial \tau_{xz}}{\partial y} + \theta \quad (83)$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{G_{yz}} \frac{\partial \tau_{yz}}{\partial x} - \theta \quad (84)$$

Subtracting Equation 84 from Equation 83

$$\frac{1}{G} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{G_{xz}} \frac{\partial^2 \phi}{\partial y^2} = -z \theta$$

or

$$\frac{G_{xz}}{G_{yz}} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -z G_{xz} \theta \quad (85)$$

To obtain boundary conditions we look at the outside of a cross-section as follows.

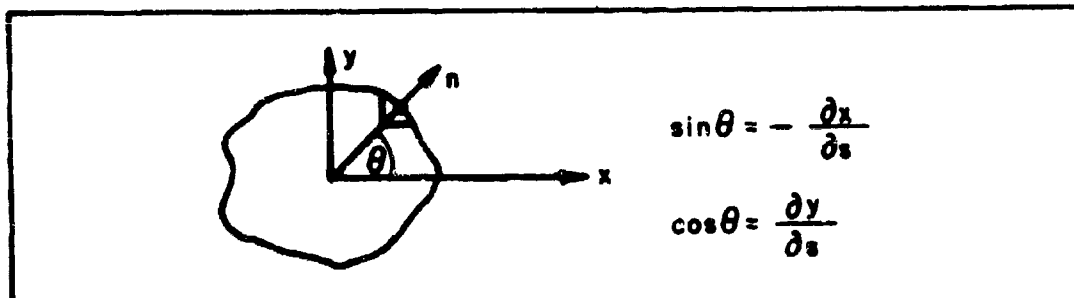


Figure 19. Cross Section of Bar Showing Boundary Forces

$$\sum F_n = 0 = \tau_{xz} \cos \theta + \tau_{yz} \sin \theta - \tau_{nz}$$

but  $\tau_{nz} = 0$  on the boundary

$$\frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial s} + \left( -\frac{\partial \phi}{\partial x} \right) \left( -\frac{\partial x}{\partial s} \right) = 0$$

or

$$\frac{d\phi}{ds} = 0$$

Take  $c = 0$  for solid cross-section, thus

$$\phi = 0 \text{ on boundary}$$

Now to find expression for torque

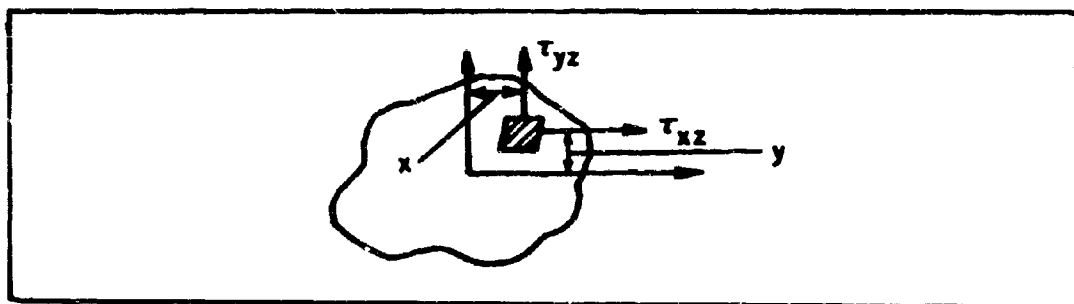


Figure 20. Cross Section Showing Stressed Element

$$dT = -y \tau_{zx} dA + x \tau_{zy} dA$$

$$T = - \int_{\text{Area}} \left( y \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} x \right) dA$$

$$= - \iint_{\text{Area}} \left( y \frac{\partial \phi}{\partial y} + \phi + x \frac{\partial \phi}{\partial x} + \phi \right) + z \iint_{\text{Area}} \phi dA$$

Green's Theorem in a plane can be used on the first integral. This theorem states (Reference 13)

$$\iint_{\text{Area}} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\text{curve}} P dx + Q dy$$

Let

$$\begin{aligned} \frac{\partial Q}{\partial x} &= x \frac{\partial \phi}{\partial x} + \phi \\ - \frac{\partial P}{\partial y} &= y \frac{\partial \phi}{\partial y} + \phi \end{aligned}$$

Then

$$Q = x\phi ; P = -y\phi$$

Now

$$\begin{aligned} \iint_{\text{Area}} \left( y \frac{\partial \phi}{\partial y} + \phi + x \frac{\partial \phi}{\partial x} + \phi \right) dx dy &= \int_{\text{Boundary}} -y\phi dx + x\phi dy \\ &= \int_{\text{Boundary}} -y\phi \frac{dx}{ds} ds + \int_{\text{Boundary}} x\phi \frac{dy}{ds} ds \end{aligned}$$

From Figure 20, the normal force at the boundary

$$\begin{aligned} \frac{dy}{ds} &= \cos \theta = n_x && \text{(x component of a unit vector in} \\ &&& \text{in the normal direction along the} \\ &&& \text{boundary)} \\ - \frac{dx}{ds} &= \sin \theta = n_y && \text{(y component of same unit vector)} \end{aligned}$$

Now

$$T = - \int_{\text{boundary}} \phi (x n_x + y n_y) ds + 2 \iint_{\text{Area}} \phi dA$$

The first integral is = 0 since  $\phi = 0$  on boundary

$$T = 2 \int \int_{\text{Area}} \phi \, dA \quad (86)$$

Figure 21 shows the cross-section of an orthotropic bar subjected to torsion. It will be assumed that  $a/b \geq 10$ , which will allow  $\tau_{xz}$  to be neglected (Reference 2). Thus the stress function  $\phi$  will be a function of  $y$  only.

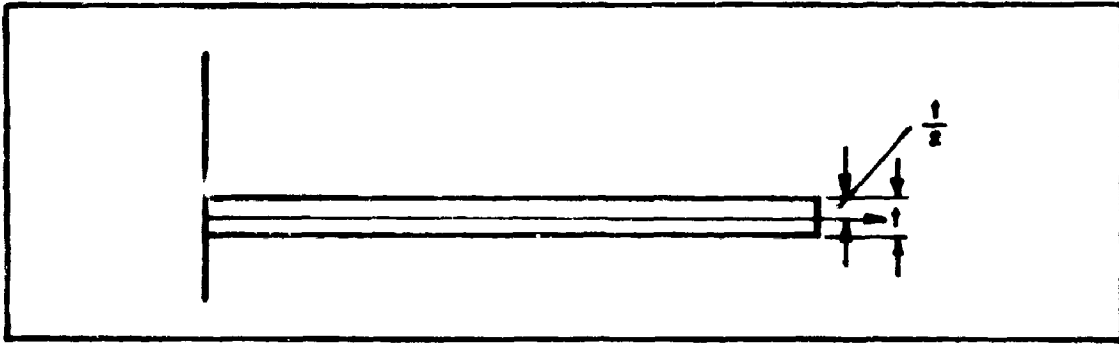


Figure 21. Cross Section of an Orthotropic Bar Subjected to Torsion

Equation 85 will become

$$\frac{d^2 \phi}{dy^2} = -2 G_{xz} \theta \quad (87)$$

Integrating Equation 87 yields

$$\phi = -G_{xz} \theta y^2 + c_1 y + c_2$$

Using the condition  $\phi = 0$  at  $y = \pm t/2$  we have

$$c_1 = 0$$

$$c_2 = G_{xz} \theta \left( \frac{t^2}{4} - y^2 \right)$$

and

$$\phi = G_{xz} \theta \left( \frac{t^2}{4} - y^2 \right) \quad (88)$$

$$\tau_{xz} = \frac{d\phi}{dy} = -2 G_{xz} \theta y \quad (89)$$

Using Equation 86 for torque

$$T = 2 \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_0^w G_{xz} \theta \left( \frac{t^2}{4} - y^2 \right) dx \, dy = \frac{wt^3 G_{xz} \theta}{3}$$

or

$$G_{xz} = \frac{3T}{w l^2 \theta} \quad (90)$$

Let us now look at the case where the cross-section of a laminate is radially isotropic (that is,  $G_{xz} = G_{yz}$ ). This is illustrated by the case of a unidirectional layup of arbitrary cross-section as shown in Figure 22.

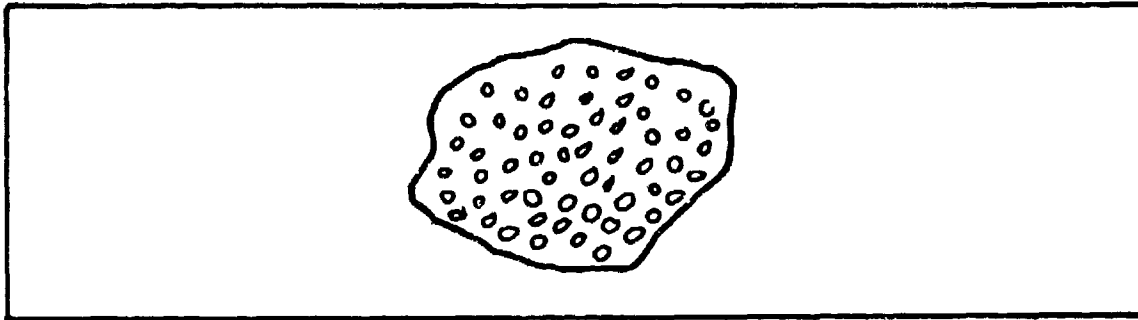


Figure 22. Unidirectional Composite with Cross Section of Arbitrary Shape

Let the torsional modulus be represented by  $G$ . Thus

$$G_{xz} = G_{yz} = G$$

and Equation 85 becomes

$$\nabla^2 \phi = -2G\theta \quad (91)$$

which is the standard torsional compatibility equation. Two cases will now be illustrated using Equation 91. The first case will be a cross-section of circular shape with radius  $R$ . Using polar coordinates let

$$\phi = \kappa (r^2 - R^2)$$

which satisfies the condition that  $\phi = 0$  on the boundary. Using this in Equation 91 yields

$$\nabla^2 \phi = 4\kappa = -2G\theta$$

Thus

$$\phi = -\frac{1}{2} G\theta (r^2 - R^2) \quad (92)$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -G\theta r \quad (93)$$

$$\tau_{yz} = - \frac{\partial \phi}{\partial x} = G \theta r \quad (94)$$

Using Equation 86

$$\begin{aligned} T &= - G \theta \int_0^R \int_0^{2\pi} (r^2 - R^2) r d\theta dr \\ &= - 2\pi G \theta \int_0^R (r^2 - R^2) r dr \\ &= - 2\pi G \theta \left( \frac{r^4}{4} - \frac{R^2 r^2}{2} \right)_0^R \end{aligned}$$

Thus

$$T = \frac{\pi G \theta R^4}{2} \quad (95)$$

or

$$G = \frac{2T}{\pi \theta R^4} \quad (96)$$

The second case we will examine is that of a rectangular bar as shown in Figure 23.

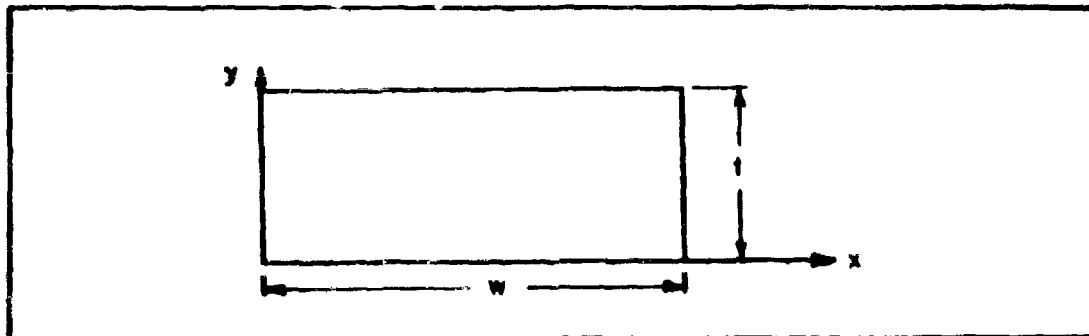


Figure 23. Rectangular Cross Section

If  $\phi$  is chosen in this case by multiplying the equations of the boundaries, the compatibility condition  $\nabla^2 \phi = 2G\theta$  cannot be satisfied. So a solution will be attempted in terms of a double Fourier Series

Let

$$\phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y$$

which satisfies  $\phi = 0$  on boundary

$$\nabla^2 \phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -a_{mn} \left[ \left( \frac{m\pi}{w} \right)^2 + \left( \frac{n\pi}{l} \right)^2 \right] \sin \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y$$

$$= -2G\theta$$

Using the equation for Fourier coefficients we have

$$a_{mn} = \frac{8G\theta}{w l \left[ \left( \frac{m\pi}{w} \right)^2 + \left( \frac{n\pi}{l} \right)^2 \right]} \int_0^l \int_0^w \sin \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y \cdot dx dy$$

Performing the integration will yield

$$a_{mn} = \frac{32G\theta l^2}{mn\pi^4 \left[ \left( \frac{m l}{w} \right)^2 + n^2 \right]} \quad (97)$$

Thus

$$\phi = \frac{32G\theta l^2}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn \left[ \left( \frac{m l}{w} \right)^2 + n^2 \right]} \sin \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y \quad (98)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial y} = \frac{32G\theta l}{\pi^3 w} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n \left[ \left( \frac{m l}{w} \right)^2 + n^2 \right]} \cos \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y \quad (99)$$

$$\tau_{xz} = \frac{\partial \phi}{\partial x} = \frac{32G\theta l}{\pi^3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m \left[ \left( \frac{m l}{w} \right)^2 + n^2 \right]} \sin \frac{m\pi}{w} x \cdot \cos \frac{n\pi}{l} y \quad (100)$$

Then

$$T = 2 \iint \phi \, dx \, dy$$

$$= \frac{64G\theta l^2}{mn\pi^4 \left[ \left( \frac{m l}{w} \right)^2 + n^2 \right]} \int_0^l \int_0^w \sin \frac{m\pi}{w} x \cdot \sin \frac{n\pi}{l} y \cdot dx \, dy$$

Let

$$T = \frac{256G\theta w t^3}{\pi^6} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n^2 \left[ \left( \frac{m t}{w} \right)^2 + n^2 \right]}$$

$$K = \frac{256}{\pi^6} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n^2 \left[ \left( \frac{m t}{w} \right)^2 + n^2 \right]} \quad (101)$$

Then

$$T = K G \theta w t^3 \quad (102)$$

Or

$$G = \frac{T}{K \theta w t^3} \quad (103)$$

Values of K have been found numerically (Reference 2) for ratios of  $t/w$  as follows

| $t/w$ | K     |
|-------|-------|
| 1     | 0.141 |
| 1.5   | 0.196 |
| 2.0   | 0.229 |
| 2.5   | 0.249 |
| 3.0   | 0.263 |
| 4.0   | 0.281 |
| 6.0   | 0.299 |
| 10.0  | 0.312 |
|       | 0.333 |

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